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On the Motion of a Charged Particle in a Uniform Electric Field with Radiation Reaction

N. D. SEN GUPTA

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-5

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Abstract

It is shown that the motion of a charged particle in a uniform electric field, obeying Dirac–Lorentz relativistic equation of motion with radiation reaction, is confined in a plane. Further, the component of velocity normal to the lines of force continuously decreases to zero. Thus, the motion asymptotically tends to a rectilinear motion along the line of force. The motion is completely described up to a correcting factor

$$1 + 0\left[\left(\frac{e^3 F}{m^2 c^4}\right)^2\right]; \qquad \frac{e^3 F}{m^2 c^4} \simeq 5.10^{-14} F$$

for electrons, F in volts cm^{-1}.

1. *Introduction*

The problem of relativistic motion of a charged particle in an external electromagnetic field taking account of reaction due to radiation has been the object of investigation since Dirac's (1938) classic paper. Recently, there has been a revival of interest in the problem due to its applications in accelerators and in astrophysics. In these cases, the particles encounter very intense fields and their energies are also very high so that usual non-relativistic approximations are no longer satisfactory. Hence, one is obliged to integrate the Dirac–Lorentz relativistic equation of motion with radiation reaction.

It is well known that the motion of a charged particle in a uniform electric field is confined to the plane which contains the initial velocity and the lines of force. This point follows clearly also in case of motion with radiation reaction in the non-relativistic approximation (Plass, 1961; Erber, 1961). Critical examination reveals that this is also the case with the Dirac–Lorentz equation of motion. Hence, the relativistic motion is also in a plane. After a preliminary discussion in Section 2 about the nature of the differential equation and the properties of the solution, we have, in Section 3, established the result.

Since the usual perturbation method is not applicable in this highly non-linear problem, we develop in Section 2 a suitable approximation method to integrate the equation of motion in the orbital plane. It is shown there that the components of the velocity normal to the lines of forces continuously decrease to zero. Hence, the motion asymptotically tends to a rectilinear one along the field direction.

2. *The Equation of Motion*

The Dirac–Lorentz relativistic equation of motion, with radiation reaction, for a charged particle moving in a uniform electric field of magnitude F along the direction $\mathbf{k}$, may be written as,

$$\dot{\mathbf{v}} - \epsilon(\ddot{\mathbf{v}} - p\mathbf{v}) = \Omega \mathbf{k} E \tag{2.1}$$

$$\dot{E} - \epsilon(\ddot{E} - pE) = \Omega \mathbf{k} . \mathbf{v} \tag{2.2}$$

Dots denote differentiation with respect to proper time τ;

$$c\mathbf{v} = \dot{\mathbf{r}}, \qquad E = \dot{t} \tag{2.3a}$$

and

$$p = \dot{\mathbf{v}} . \dot{\mathbf{v}} - \dot{E}^2 \tag{2.3b}$$

E is the relativistic energy of the particle in unit of its rest energy mc^2. Further,

$$\epsilon = \frac{2e^2}{3mc^3}, \qquad \Omega = \frac{eF}{mc} \tag{2.4}$$

Before proceeding to integrate equation (2.1), let us make some observations which are pertinent to the physical and mathematical nature of the problem. In equation (2.1) there are two parameters, namely ϵ and Ω; both of them contain e, m and c. Ω is inseparable from the incident field. ϵ is attributed to the radiation reaction and $\epsilon = 0$ means that radiation reaction is neglected. We are interested in the solutions of the equation of motion which are meaningful as $\epsilon \to 0$, irrespective of Ω. Hence, we cannot impose arbitrary initial acceleration. It can be easily verified that the only solutions which possess this property are those for which $\dot{\mathbf{v}}(t)$ as $\epsilon \to 0$ at t_0 is the same as obtained from equation (2.1) with $\epsilon = 0$ and given initial $\mathbf{v}(t_0)$ and $\mathbf{r}(t_0)$. Though our equation of motion is of third order by this prescription, we are still in the realm of Newtonian mechanics, in so far as the motion is uniquely determined when initial velocity and acceleration are known. This is due to the fact that as $\epsilon \to 0$, the order of equation (2.1) is reduced. Our procedure is different from that of Bhabha (1946), according to which physical solutions are only those which can be continued to $e \to 0$. In this limit the incident field also disappears and $\dot{\mathbf{v}}(t_0) = 0$ as $e \to 0$ which seriously restricts the solutions. It must be emphasised that by this method one is not seeking a solution for ϵ small, but solutions which are regular as $\epsilon \to 0$. From the viewpoint of the theory of differential equations $\epsilon = 0$ is a singular point of equation (2.1) and we are seeking only those solutions which are regular as $\epsilon \to 0$. The existence of such solutions specially for linear differential equations have been investigated by Poincaré (1895) and recently by Tihonov (1948) and Gradstein (1950). It is relevant to mention that according to this prescription the motion in absence of external field is with uniform velocity, as self-acceleration no longer appears.

3. *The Orbital Plane*

From equation (2.1) it follows that

$$\mathbf{k}\times\mathbf{v}.\dot{\mathbf{v}} - \epsilon\mathbf{k}\times\mathbf{v}.\ddot{\mathbf{v}} = 0$$

therefore,

$$\exp(-\tau/\epsilon)(\mathbf{k}\times\mathbf{v}.\dot{\mathbf{v}}) = \text{constant} \tag{3.1}$$

But from equation (2.1) $\dot{\mathbf{v}}\times\mathbf{k} = 0$ as $\epsilon \to 0$, the constant is zero. Thus

$$\mathbf{k}\times\mathbf{v}.\dot{\mathbf{v}} = 0 \tag{3.2}$$

Hence, the motion is confined to the plane which contains the initial velocity and the lines of forces, as in the absence of radiation reaction. This is also expected from the fact that the radiation emitted by such a charged particle is symmetric with respect to the two sides of this plane, so that the resultant radiation reaction normal to this plane is zero. It follows further that if the initial velocity is along the lines of force the motion is along the line of force also. This special case has been discussed by Dirac (1938).

Without any loss of generality we can take the initial velocity along $\mathbf{k}$ to be zero; if it is not zero we can pass to the frame in which it is zero and the incident field is still purely electric. It may be mentioned that the above integral is valid even if F depends on $\mathbf{k}\times\mathbf{r}$ but is along $\mathbf{k}$. The analogous integral in the case of motion in a magnetic field is that the velocity along the lines of force is constant.

4. *The Motion in the Orbital Plane*

Let $\mathbf{j}\mathbf{v}_0 c$ be the initial velocity at $t=0$, which by choice of reference system is perpendicular to $\mathbf{k}$. The equation of motion may be conveniently expressed in terms of

$$\left.\begin{aligned} P_+ = E+\mathbf{k}.\mathbf{v}, \qquad P_- = E-\mathbf{k}.\mathbf{v} \\ \mathbf{j}.\mathbf{v} = v_\perp \end{aligned}\right\} \tag{4.1}$$

and

$$\dot{P}_+ - \epsilon\ddot{P}_+ + P_+(\epsilon p - \Omega) = 0 \tag{4.2}$$

$$\dot{P}_- - \epsilon\ddot{P}_- + P_-(\epsilon p + \Omega) = 0 \tag{4.3}$$

$$\dot{v}_\perp - \epsilon\ddot{v}_\perp + \epsilon p v_\perp = 0 \tag{4.4}$$

$$p - \frac{\epsilon}{2}\dot{p} = \frac{\Omega}{2}(\dot{P}_+P_- - \dot{P}_-P_+) \tag{4.5}$$

Next, we observe that the characteristic time concomitant to radiation reaction is $\epsilon \simeq 10^{-23}$ sec, but Ω which has the dimension of T^{-1} (electric analogue of Larmor frequency) is very small in comparison with ϵ^{-1}. For fairly high-intensity field $E \simeq 10^6$ V cm^{-1} $\Omega \simeq 5 \times 10^{15}$ sec^{-1}, so that $\epsilon\Omega \simeq 3 \times 10^{-8}$. Thus it is quite justifiable to consider $\epsilon\Omega$ as small and seek approximate solutions whose accuracy may be successively increased in ascending powers of $\epsilon\Omega$.

Since p in equations (4.2) and (4.3) appears with coefficient ϵ for the first-order term, we can substitute the value of p obtained from equation (4.5) by putting $\epsilon = 0$. Thus

$$p = \Omega^2(1 + v_\perp^2) \tag{4.6}$$

Further, from equation (4.4) in this order $v_\perp$ is constant and

$$1 + v_\perp^2 = E_0^2 \tag{4.7}$$

On integrating equations (4.2) and (4.3) with p given by equation (4.6) one obtains†

$$P_+ = \exp[g(\Omega)\,\tau]\,E_0, \qquad P_- = \exp[g(-\Omega)\,\tau]\,E_0 \tag{4.8}$$

where

$$g(\Omega) = \frac{\Omega(1 - \epsilon\Omega E_0^2)}{1 - \epsilon\Omega} \tag{4.9}$$

and

$$j.\mathbf{v} = \exp(-\epsilon\Omega^2 E_0^2\,\tau)\,v_0\,E_0 \tag{4.10}$$

This solution may be improved if one starts with the value of p obtained from equation (4.9). One obtains

$$P_+ = \exp[f(\Omega, \tau)]\,E_0, \qquad P_- = \exp[f(-\Omega, \tau)]\,E_0 \tag{4.11}$$

where

$$f(\Omega, \tau) = \frac{1}{1 - \epsilon\Omega}\left[\Omega\tau + \frac{E_0^2\{\exp[-2\epsilon\Omega^2(E_0^2 - 1)\,\tau] - 1\}}{2(E_0^2 - 1)}\right] \tag{4.12}$$

$$v_\perp = v_0\,E_0 . \exp - [\epsilon\Omega^2 E_0^2\,\tau - v_0^2 E_0^2 \exp(-\epsilon\Omega^2 E_0^2\,\tau) \sinh \epsilon\Omega^2 E_0\,\tau] \times$$
$$\times (1 - \epsilon^2\Omega^2 E_0^2) \tag{4.13}$$

Hence, it only introduces a correcting factor $1 + 0(\epsilon^2\Omega^2)$, and the first-order solution is valid to a very good degree of approximation. It may be noted that for $E_B \simeq 6 \times 10^9$ V cm^{-1} (which is the field due to the proton at the first Bohr radius), $\epsilon^2\Omega_B{}^2 \simeq 9 \times 10^{-8}$; E_B is the threshold field for quantum effect.

4.1 *The Longitudinal Motion*

From equations (4.8) and (4.9)

$$\mathbf{k}.\mathbf{v} = E_0 \exp[-\epsilon\Omega^2(E_0^2 - 1)\tau] \sinh \Omega\tau \tag{4.1.1}$$

$$E = E_0 \exp[-\epsilon\Omega^2(E_0^2 - 1)\tau] \cosh \Omega\tau \tag{4.1.2}$$

† In order to obtain equations (4.8–4.9), it is assumed that $v_0^2 > 1$. Otherwise the expressions for $P_\pm$ and $v_\perp$ become extremely involved. It may be emphasized that the basic nature of the motion as described subsequently remains the same without this approximation.

and

$$\frac{d}{dt}\mathbf{k}\cdot\mathbf{r} = c\tanh\Omega\tau \tag{4.1.3}$$

on further integration

$$\mathbf{k}\cdot\mathbf{r} = \frac{eE_0}{\Omega}\exp\left[-\epsilon\Omega^2(E_0{}^2-1)\tau\right]\{(\cosh\Omega\tau-1)+\epsilon\Omega(E_0{}^2-1)\sinh\Omega\tau\} \tag{4.1.4}$$

$$t = \frac{E_0}{\Omega}\exp[-\epsilon\Omega^2(E_0{}^2-1)\tau]\{\sinh\Omega\tau+\epsilon\Omega(E_0{}^2-1)(\cosh\Omega\tau-1)\} \tag{4.1.5}$$

Since τ increases monotonically with t, the longitudinal velocity $(d/dt)\mathbf{k}\cdot\mathbf{r}$ and E monotonically increases and asymptotically tends to c and ∞, respectively.

4.2 *The Transverse Motion*

From equations (4.10) and (4.1.2), the transverse velocity is given by

$$\frac{d}{dt}\mathbf{j}\cdot\mathbf{r} = cv_0\exp(-\epsilon\Omega^2\tau)\operatorname{sech}\Omega\tau \tag{4.2.1}$$

and

$$\mathbf{j}\cdot\mathbf{r} = \frac{v_0\,c}{\epsilon\Omega^2 E_0}[1-\exp(-\epsilon\Omega^2 E_0{}^2\tau)] \tag{4.2.2}$$

Hence, the velocity in the transverse direction monotonically decreases to zero. After a short interval, $\simeq E_0/\Omega$, the transverse velocity is reduced to $v_0/2$ and this interval decreases with increase of field intensity, for $F\simeq 1$ V cm^{-1}, $1/\Omega\simeq 2\times 10^{-10}$ sec. This leads to the conclusion that the motion in a field which is not very weak quickly attains its asymptotic property; namely, rectilinear motion along the lines of force, so that most of the radiation emitted by the particle is at the cost of energy acquired from the field. The total displacement in this direction is given by

$$\mathbf{j}\cdot\mathbf{r}(\infty) = \frac{v_0}{\epsilon\Omega^2 E_0} \tag{4.2.3}$$

5. *Discussion*

The method of successive approximations followed here is quite different from the usual perturbation expansions in ascending powers of ϵ, which are valid only for a short interval of time. On the other hand, our expressions are valid for all time and are accurate up to a factor of $1+0(\epsilon^2\Omega^2)$. The error involved is practically negligible for almost all attainable field intensities, and the conclusions are valid so long as quantum effects are not introduced. Again, the approximation used is quite different from the non-relativistic approximation which starts from $p = 0$.

Before we conclude, we want to make a few remarks about the nature of the radiation emitted. Since the acceleration is known, the formal expression for the radiation field can be obtained directly. But one notes that, due to the similarity of the expression (4.1.3) with that in non-relativistic approximation, the basic nature of the radiation emitted, which is known, remains the same except for small corrections, depending on the parameter $\epsilon\Omega$.

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